# Partial derivatives

Let the function $f(x, y) = x^2 + y^2$

1. Applying the definition, prove that it is differentiable at the origin (the tangent plane is a good approximation of the function at the point).
2. Prove that it is differentiable over the entire $\mathbb{R}^2$ by applying the sufficient condition for differentiability.

## Solution

1. First, we check if the function is differentiable at the point. To do this, we obtain the derivatives

$$f'x = 2x$$

$$f'y = 2y$$

The derivatives evaluated at the point are equal to 0.

Next, we check if the function is continuous at the point, which is also true since the limit exists and the function at the point, and they are equal to each other. Lastly, we check if the derivatives are continuous at the point, which is also satisfied since the derivatives at the point are 0, and the limit at the point is also 0.
By definition:

$$\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{\Delta z} = f'_x(x_0, y_0)\Delta x + f'_y(x_0, y_0)\Delta y + \epsilon(\Delta x, \Delta y)\sqrt{\Delta x^2 + \Delta y^2}$$

then, if

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \epsilon(\Delta x, \Delta y) = 0$$

the function is differentiable.

In this case:

$$\Delta z = (0 + \Delta x)^2 + (0 + \Delta y)^2 - 0 = 0\Delta x + 0\Delta y + \epsilon(\Delta x, \Delta y)\sqrt{\Delta x^2 + \Delta y^2}$$

$$\frac{\Delta x^2 + \Delta y^2}{\sqrt{\Delta x^2 + \Delta y^2}} = \epsilon(\Delta x, \Delta y)$$

$$\sqrt{\Delta x^2 + \Delta y^2} = \epsilon(\Delta x, \Delta y)$$

We let $\Delta y$ and $\Delta x$ tend to 0:

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \sqrt{\Delta x^2 + \Delta y^2} = 0$$

2. The sufficient condition is met if the partial derivatives are continuous at any point, then the function is differentiable at any point.

**In this case, the partial derivatives are polynomials and therefore continuous at any point.**